HIGGS VARIETIES AND FUNDAMENTAL GROUPS

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ABSTRACT. After reviewing some “fundamental group schemes” that can be attached to a variety by means of Tannaka duality, we consider the example of the Higgs fundamental group scheme, surveying its main properties and relations with the other fundamental groups, and giving some examples.

1. INTRODUCTION

As the usual fundamental group is not well suited to study schemes equipped with the Zariski topology, Grothendieck introduced in [11] the so-called *étale fundamental group*. The usual fundamental group of a space X may be regarded as the group of deck transformations of the universal covering of X. Heuristically, one replaces covering spaces by finite étale covers; however, in this case there is no universal object, so that one needs to take an inverse limit. More technically, given a scheme X, which one assumes to be connected and locally noetherian, and after fixing a geometric point x of X, one considers the set of

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pairs (p, y), where $p\colon Y \to X$ is a finite étale cover, and $y \in Y$ is a geometric point such that $p(y) = x$. The set I of such pairs is partially ordered by the relation $(p, y) \geq (p', y')$ if there is a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "p"'] & Y' \arrow[d, "p'"] \\ & X \end{tikzcd}$$

with $y' = f(y)$. Then one defines

$$\pi_1^{\text{ét}}(X, x) = \varprojlim_{i \in I} \operatorname{Aut}_X(p_i, y_i).$$

If X is a scheme of finite type over $\mathbb{C}$, the étale fundamental group $\pi_1^{\text{ét}}(X, x)$ is the profinite completion of the topological fundamental group $\pi_1(X, x)$.

In spite of the naturalness of its definition, the étale fundamental group, for a field of positive characteristic, fails to enjoy some quite reasonable properties; for instance, it is not a birational invariant, and is not necessarily zero for rational varieties. To circumvent these undesirable features, M.V. Nori introduced another kind of "fundamental group" (which coincides with the étale fundamental group for fields of characteristic zero) [23]. One of the properties that make this fundamental group particularly interesting is that it is introduced in terms of the so-called *Tannaka duality.* Nori considered, on a scheme X over a field $\Bbbk$, vector bundles E having the following property: there exists a Γ-torsor P on X, where Γ is a finite group, such that the pullback of E to P is trivial. Such vector bundles are said to be *essentially finite*. The category of essentialy finite vector bundles on X, with the functor to the category $\mathbf{Vect}_{\Bbbk}$ of finite-dimensional vector spaces over $\Bbbk$ given by $E \mapsto E_x$, where E_x is the fibre over a fixed geometric point x of X, is an example of a *neutral Tannakian category* over $\Bbbk$. Any such category is equivalent to the category of representations of a group scheme over $\Bbbk$ ([9], and see Section 2.2). This group scheme is by definition Nori's fundamental group scheme $\pi_1^N(X, x)$.

Two more "fundamental groups" have been associated with a variety in terms of Tannaka duality. Langer [13, 14] considered the category of numerically flat vector bundles (i.e., vector bundles that are numerically effective together with their duals). The associated fundamental group scheme was denoted by $\pi_1^S(X, x)$ (this group was introduced in the case of curves also in [3]). For varieties over the complex numbers, Simpson considered the category of semi-harmonic bundles, i.e., semistable Higgs vector bundles with vanishing rational Chern classes [24, 25]. The associated fundamental group scheme was denoted $\pi_1^{\mathrm{alg}}(X,x)$; quite interestingly, it is the proalgebraic completion of the topological fundamental group. All these fundamental groups are related by morphisms according to the scheme

$$\pi_1^{\mathrm{alg}}(X,x) \twoheadrightarrow \pi_1^{S}(X,x) \twoheadrightarrow \pi_1^{N}(X,x) \twoheadrightarrow \pi_1^{\text{ét}}(X,x)$$

where each arrow is a faithfully flat morphism.

In [4, 5] we introduced notions of numerical effectiveness and numerical flatness for Higgs bundles. The definition of these notions stems from the remark that the universal quotient bundles over the Grassmann bundles $\mathrm{Gr}_s(E)$ of a numerically effective vector bundle are numerically effective (recall that the sections of the Grassmann bundle $\mathrm{Gr}_s(E) \to X$ of a vector bundle E on X are in a one-to-one correspondence with rank s locally free quotients of E). Given a Higgs vector bundle $\mathfrak{E} = (E, \phi)$, we consider closed subschemes $\mathfrak{Gr}_s(\mathfrak{E}) \subset \mathrm{Gr}_s(E)$ that analogously parameterize locally free Higgs quotients of $\mathfrak{E}$. Then $\mathfrak{E}$ is said to be H-numerically effective if the universal Higgs quotients on $\mathfrak{Gr}_s(\mathfrak{E})$ are H-numerically effective, according to a definition which is recursive on the rank. Finally, a Higgs bundle is said to be H-numerically flat if $\mathfrak{E}$ and its dual Higgs bundle $\mathfrak{E}^*$ are H-numerically effective. H-numerically flat Higgs bundles make up again a neutral Tannakian category; the corresponding group scheme is denoted by $\pi_1^H(X,x)$ [2].

Numerically flat vector bundles, if equipped with the zero Higgs field, are H-numerically flat, so that there is a faithfully flat morphism $\pi_1^H(X,x) \twoheadrightarrow \pi_1^S(X,x)$. The relation of $\pi_1^H(X,x)$ with Simpson's proalgebraic fundamental group $\pi_1^{\mathrm{alg}}(X,x)$ is more subtle (see Section 2.2). Again, since semi-harmonic bundles are H-numerically flat, there is a faithfully flat morphism $\pi_1^H(X,x) \twoheadrightarrow \pi_1^{\mathrm{alg}}(X,x)$. The fact that the groups may be isomorphic is related with a conjecture about the so-called *curve semistable Higgs bundles* — i.e., Higgs bundles that are semistable after pullback to any smooth projective curve [5, 8, 15]. The main purpose of this note is to gather and briefly discuss what is presently known about this question.

In the preliminary Section 2 we recall the definitions and main properties of the profinite and proalgebraic completions of discrete groups, and the definition of Nori's, Langer's and Simpson's fundamental groups. Section 3 reviews the introduction of the Higgs fundamental group. The final Section 4 provides some examples.

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2. Preliminaries

2.1. **Profinite and proalgebraic completions.** We recall here the construction of profinite and proalgebraic completions of groups. They are inverse limits of systems of finite and algebraic groups, respectively. They enjoy natural universal properties, and in both cases there is a natural homomorphism from the group to its completion with dense image. References for the two constructions are [18] and [1].

Definition 2.1. *A profinite group is a topological group which is the inverse limit of an inverse system of discrete finite groups. The profinite completion $\hat{G}$ of a group G is the inverse limit of the system formed by the quotient groups G/N of G, where N are normal subgroups of G of finite index, partially ordered by inclusion.*

For instance, the profinite completion of $\mathbb{Z}$ is

$$\hat{\mathbb{Z}} = \prod_p \mathbb{Z}(p)\,,$$

where p runs over the prime numbers, and $\mathbb{Z}(p)$ is the ring of p-adic integers [18].

Every finite group, equipped with the discrete topology, is profinite, and therefore coincides with its profinite completion.

Definition 2.2. *A proalgebraic group over $\mathbb{k}$ is the inverse limit of a system of algebraic groups over $\mathbb{k}$. A proalgebraic completion of a discrete group Γ consists of a proalgebraic group $A(\Gamma)$ over $\mathbb{k}$ with a homomorphism $\rho : \Gamma \to A(\Gamma)$ satisfying the following universal property: for any proalgebraic group H and any homomorphism $\rho_H : \Gamma \to H$ there exists a unique morphism $f : A(\Gamma) \to H$ such that $\rho_H = f \circ \rho$.*

The universal property in Definition 2.2 ensures that a proalgebraic completion for Γ is unique up to unique isomorphism. The image of ρ is Zariski dense in $A(\Gamma)$.

One way of constructing the proalgebraic completion $A(\Gamma)$ is to take the closure of the diagonal image of Γ into the product of all the images of all finite dimensional $\mathbb{k}$-representations of Γ. Another construction is obtained via Tannaka duality (see Section 2.2), as the tensor product preserving automorphisms of the forgetful functor from the category of finite dimensional Γ-modules to the category of finite dimensional $\mathbb{k}$-vector spaces.

When Γ is abelian, its proalgebraic completion $A(\Gamma)$ is quite easily described as a direct product [1, Example 2]

$$A(\Gamma) = U(\Gamma) \times T \times \hat{\Gamma} \tag{1}$$

where $U(\Gamma) = \mathrm{Hom}(\Gamma, \Bbbk)^\vee$, while T is a protorus (i.e., the inverse limit of a system of tori) whose group of characters is the torsion-free quotient of $\mathrm{Hom}(\Gamma, \Bbbk^*)$, and $\hat{\Gamma}$ is the profinite completion of Γ. In particular, when $\Gamma = \mathbb{Z}$, then $U(\mathbb{Z}) = \mathbb{G}_a(\Bbbk)$ and the character group of T is $\mathbb{G}_m(\Bbbk)$:

$$A(\mathbb{Z}) = \mathbb{G}_a(\Bbbk) \times T \times \hat{\mathbb{Z}}. \tag{2}$$

2.2. **Fundamental groups.** Nori's notion of fundamental group scheme may be seen as a powerful generalization of the well-known fact that flat vector bundles on (say) a differentiable manifold X correspond to representations of the topological fundamental group of X. The generalization is made in terms of *Tannaka duality*, i.e., the fact that the categories of representations of an affine group scheme can be characterized as abelian tensor categories satisfying some suitable conditions [9].

We recall that an abelian $\Bbbk$-linear tensor category $\mathbf{C}$ is *rigid* [9, Def. 1.7] if

- Hom and $\otimes$ satisfy a distributive property over finite families, i.e., for any pair of finite families $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ of objects in $\mathbf{C}$ the morphisms
$$\otimes_{i\in I} \mathrm{Hom}(A_i, B_i) \to \mathrm{Hom}(\otimes_{i\in I} A_i, \otimes_{i\in I} B_i)$$
are isomorphisms;
- all objects are reflexive, i.e., the natural map to their double dual is an isomorphism.

A *neutral Tannakian category over a field* $\Bbbk$ is a rigid abelian $\Bbbk$-linear tensor category $\mathbf{C}$ together with a faithful exact $\Bbbk$-linear tensor functor $\omega\colon \mathbf{C} \longrightarrow \mathbf{Vect}_\Bbbk$, where $\mathbf{Vect}_\Bbbk$ is the category of $\Bbbk$-vector spaces, and ω is called the *fibre functor.* Then, there exists an affine group scheme G over $\Bbbk$ such that $\mathbf{C}$ is equivalent to the category $\mathbf{Rep}_\Bbbk(G)$ of $\Bbbk$-linear representations of G.

Given a scheme over a field $\Bbbk$, Nori introduced his fundamental group scheme $\pi_1^N(X, x)$ as the affine group scheme representing the Tannakian category of *essentially finite vector bundles:* a vector bundle E on X is said to be essentially finite if there is a Γ-torsor P over X (where Γ is a finite group) such that the pullback of E to P is trivial [23, 22]. When $\Bbbk$ has characteristic 0, $\pi_1^N(X, x)$ coincides with Grothendieck's étale fundamental group [11], which is isomorphic to the profinite completion of the topological fundamental group. In positive characteristic Nori's group gives an improvement with respect to the étale group, as the latter does not take into account inseparable covers.

Another kind of fundamental group scheme was introduced by Langer [13, 14]. Given a smooth variety X over an algebraically closed field $\Bbbk$, one considers *numerically effective vector bundles*, i.e., vector bundles E on X such that the hyperplane bundle of the projectivization $\mathbb{P}E$ of E is numerically effective. Moreover, a vector bundle E is *numerically flat* if both E and its dual bundle E^* are numerically effective. The category $\mathbf{NF}(X)$ of numerically flat vector bundles on X is a neutral Tannakian category, with a fibre functor which takes a numerically flat vector bundle E to its fibre E_x at x. Then $\pi_1^S(X,x)$ is the affine group scheme representing the Tannakian category $\mathbf{NF}(X)$.

These various fundamental groups are related by morphisms according to the scheme

$$\pi_1^S(X,x) \twoheadrightarrow \pi_1^N(X,x) \twoheadrightarrow \pi_1^{\text{ét}}(X,x).$$

These morphisms are actually faithfully flat. Moreover, they are isomorphisms when $\Bbbk$ has characteristic zero.

In the case $\Bbbk = \mathbb{C}$ Simpson also introduced the *algebraic fundamental group* $\pi_1^{\text{alg}}(X,x)$, which is the proalgebraic completion of the topological fundamental group. It is Tannaka dual to the category of semistable Higgs bundles on X with vanishing Chern classes (see next Section). Since numerically flat vector bundles are semistable and have vanishing Chern classes [10], there is a natural morphism $\pi_1^{\text{alg}}(X,x) \twoheadrightarrow \pi_1^S(X,x)$, which is again a faithfully flat morphism.

2.3. **Higgs bundles and the algebraic fundamental group.** Let X be a smooth projective variety of dimension n over an algebraic closed field $\Bbbk$ of characteristic zero, and denote by Ω^1_X the cotangent bundle of X. Let L be a very ample line bundle on X, and denote by H its numerical class. The degree of a coherent $\mathcal{O}_X$–module F is defined as

$$\deg F = c_1(F) \cdot H^{n-1},$$

and if $\operatorname{rk} F \neq 0$, one defines the *slope* of F to be

$$\mu(F) = \frac{\deg F}{\operatorname{rk} F}.$$

Definition 2.3. *A Higgs sheaf $\mathfrak{E}$ on X is a pair (E,ϕ), where E is a torsion-free coherent sheaf on X and*

$$\phi \colon E \longrightarrow E \otimes \Omega^1_X$$

is a homomorphism of $\mathcal{O}_X$*-modules such that* $\phi \wedge \phi = 0$*. A Higgs subsheaf of a Higgs sheaf* $\mathfrak{E} = (E, \phi)$ *is a pair* (F, ϕ')*, where* F *is a subsheaf of* E *such that* $\phi(F) \subset F \otimes \Omega^1_X$*, and* $\phi' = \phi|_F$*. A Higgs bundle is a Higgs sheaf* $\mathfrak{E}$ *such that* E *is a locally free* $\mathcal{O}_X$*-module. If* $\mathfrak{E} = (E, \phi)$ *and* $\mathfrak{F} = (F, \psi)$ *are Higgs sheaves, a morphism* $f \colon \mathfrak{E} \longrightarrow \mathfrak{F}$ *is a homomorphism of* $\mathcal{O}_X$*-modules* $f \colon E \longrightarrow F$ *such that the diagram*

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \psi} \\ E \otimes \Omega^1_X & \xrightarrow{f \otimes \mathrm{id}} & F \otimes \Omega^1_X \end{array}$$

commutes.

Definition 2.4. *A Higgs sheaf* $\mathfrak{E} = (E, \phi)$ *is semistable (respectively, stable) if* $\mu(F) \leq \mu(E)$ *(respectively,* $\mu(F) < \mu(E)$*) for every Higgs subsheaf* (F, ϕ') *of* $\mathfrak{E}$ *with* $0 < \operatorname{rk} F < \operatorname{rk} E$*.*

As mentioned earlier, the category which has semistable Higgs bundles with vanishing Chern classes on X as objects, and morphisms of Higgs sheaves as morphisms, is a neutral Tannakian category $\mathbf{SH}(X)$ (the fibre functor is the usual one).[1] The algebraic fundamental group $\pi_1^{\mathrm{alg}}(X, x)$ is the group scheme representing this category. In [24] Simpson proves that $\pi_1^{\mathrm{alg}}(X, x)$ is actually the proalgebraic completion of the fundamental group $\pi_1(X, x)$ of X.

3. Higgs fundamental groups

3.1. **Numerically flat Higgs bundles.** We want to introduce one more "fundamental group" defined in terms of Higgs bundles. We shall introduce another Tannakian category and the fundamental group scheme representing it. This category has as objects a certain class of Higgs bundles satisfying a property that we call H-numerical effectiveness.

Let us start by recalling the definition of numerical effective ordinary vector bundles on a projective variety X.

[1]The notation for this category comes from the fact that Simpson called these Higgs bundles *semi-harmonic* [25].

Definition 3.1. (i) *A line bundle L on X is said to be numerically effective (nef for short) if, for every pair (C, f), where C is a smooth projective irreducible curve and $f\colon C \longrightarrow X$ is a morphism, the line bundle f^*L on C has nonnegative degree.*

(ii) *A vector bundle E is numerically effective if the hyperplane line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on the projectivization $\mathbb{P}(E)$ of E is numerically effective.*

The main properties of numerically effective vector bundles are described in [12, 16, 17].

In order to give the definition of Higgs numerical flatness it is necessary to introduce the notion of Higgs Grassmann bundle. Let E be a vector bundle of rank r on X, and let $s < r$ be a positive integer. The Grassmann bundle $\mathrm{Gr}_s(E)$ on X parameterizes quotients of fibres of E of dimension s. Let $p_s : \mathrm{Gr}_s(E) \longrightarrow X$ be the natural projection, then there exists a universal short exact sequence

$$0 \longrightarrow S_{r-s,E} \xrightarrow{\psi} p_s^*E \xrightarrow{\eta} Q_{s,E} \longrightarrow 0 \tag{3}$$

of vector bundles on $\mathrm{Gr}_s(E)$, with $S_{r-s,E}$ the rank $r-s$ universal subbundle and $Q_{s,E}$ the rank s universal quotient. Given now a Higgs bundle $\mathfrak{E} = (E, \phi)$, we have the closed subschemes $\mathfrak{Gr}_s(\mathfrak{E}) \subset \mathrm{Gr}_s(E)$ parameterizing rank s locally free Higgs quotients, i.e., locally free quotients of E whose corresponding kernels are ϕ-invariant. In other words, $\mathfrak{Gr}_s(\mathfrak{E})$ *(the Grassmannian of locally free rank s Higgs quotients of $\mathfrak{E}$)* is the closed subscheme of $\mathrm{Gr}_s(E)$ defined by the vanishing of the composed morphism

$$(\eta \otimes \mathrm{Id}) \circ p_s^*(\phi) \circ \psi\colon S_{r-s,E} \longrightarrow Q_{s,E} \otimes p_s^*\Omega^1_X .$$

Let $\rho_s := p_s|_{\mathfrak{Gr}_s(\mathfrak{E})} : \mathfrak{Gr}_s(\mathfrak{E}) \longrightarrow X$ be the induced projection. The restriction of (3) to $\mathfrak{Gr}_s(\mathfrak{E})$ provides the universal exact sequence

$$0 \longrightarrow \mathfrak{S}_{r-s,\mathfrak{E}} \xrightarrow{\psi} \rho_s^*\mathfrak{E} \xrightarrow{\eta} \mathfrak{Q}_{s,\mathfrak{E}} \longrightarrow 0, \tag{4}$$

where $\mathfrak{Q}_{s,\mathfrak{E}} := Q_s|_{\mathfrak{Gr}_s(\mathfrak{E})}$ is equipped with the quotient Higgs field induced by the Higgs field $\rho_s^*\phi$. The universal property satisfied by $\mathfrak{Gr}_s(\mathfrak{E})$ is that given any morphism of $\mathbb{k}$-varieties $f : T \to X$, f factors through $\mathfrak{Gr}_s(\mathfrak{E})$ if and only if the pullback $f^*(E)$ admits a rank s Higgs quotient. In that case the pullback of the above universal sequence on $\mathfrak{Gr}_s(E)$ gives the desired quotient of $f^*(E)$.

Definition 3.2. *A rank one Higgs bundle $\mathfrak{E}$ is said to be Higgs-numerically effective (H-nef for short) if it is numerically effective in the usual sense. If* $\operatorname{rk}\mathfrak{E} \geq 2$*, we inductively define H-nefness by requiring that*

(i) *all Higgs bundles* $\mathfrak{Q}_{s,\mathfrak{E}}$ *are Higgs-nef (see* (4)*), and*

(ii) *the determinant line bundle* $\det(E)$ *is nef.*

If both $\mathfrak{E}$ *and* $\mathfrak{E}^*$ *are Higgs-numerically effective,* $\mathfrak{E}$ *is said to be Higgs-numerically flat (H-nflat).*

Definition 3.2 immediately implies that the first Chern class of an H-numerically flat Higgs bundle is numerically equivalent to zero.

3.2. **The Higgs fundamental group.** Given a smooth projective variety X over a field $\Bbbk$ of characteristic zero, we consider the category $\mathbf{HNF}(X)$ whose objects are H-numerically flat Higgs bundles on X, and morphisms are morphisms of Higgs sheaves. By Proposition 3.7 in [2] kernels and cokernels in these categories are locally free, which implies that $\mathbf{HNF}(X)$ is an abelian category. Another important property is the fact that the tensor product of H-nef Higgs bundles is H-nef [2]. The proof of this is based on the existence of the Harder-Narasimhan filtration for semistable Higgs bundles, the fact that the pullback by a surjective morphism of a Higgs bundle is H-nef if and only if the Higgs bundle is H-nef, and that every Higgs quotient of an H-nef Higgs bundle is H-nef. So we have

Theorem 3.3. *The category* $\mathbf{HNF}(X)$*, with the usual fibre functor, is a neutral Tannakian category.*

Definition 3.4. [2, Def. 4.2] *Let* $x \in X$*. The* Higgs fundamental group scheme $\pi_1^H(X,x)$ *is the affine group scheme representing the category* $\mathbf{HNF}(X)$*.*

The natural inclusion $\mathbf{NF}(X) \hookrightarrow \mathbf{HNF}(X)$ induces a faithfully flat homomorphism of group schemes $\pi_1^H(X,x) \twoheadrightarrow \pi_1^S(X,x)$.

We recall from [2] some properties of the fundamental Higgs scheme.

- If $f\colon X' \longrightarrow X$ is a faithfully flat morphism of projective varieties over $\Bbbk$, with $f_*\mathcal{O}_{X'} \simeq \mathcal{O}_X$ and $f(x') = x$, then the induced morphism $\pi_1^H(X',x') \longrightarrow \pi_1^H(X,x)$ is a faithfully flat morphism.
- If $\pi_1^H(X,x) = \{e\}$, the category $\mathbf{HNF}(X)$ is equivalent to the category $\mathbf{Vect}_\Bbbk$ of finite-dimensional vector spaces, so that all H-nflat Higgs bundles on X are trivial.
- If the natural morphism $\pi_1^H(X,x) \to \pi_1^S(X,x)$ is an isomorphism, the categories $\mathbf{HNF}(X)$ and $\mathbf{NF}(X)$ are equivalent, so that all H-nflat Higgs bundles on X have zero Higgs field.

- If X, Y are projective varieties over $\Bbbk$, and x, y are points in X, Y, respectively, there is a naturally defined morphism

$$\pi_1^H(X \times_{\Bbbk} Y, (x, y)) \longrightarrow \pi_1^H(X, x) \times \pi_1^H(Y, y).$$

3.3. **Higgs varieties.** The Higgs fundamental scheme is related to a conjecture, which was established in [5] (see [15] for a review), about an extension of a result by [21] which generalizes to higher dimensions Miyaoka's semistability criterion for bundles on curves [19].

Let X be a smooth complex projective variety with a polarization H. If E is a vector bundle on X we denote by $\Delta(E) \in H^4(X, \mathbb{Q})$ its discriminant, i.e., the characteristic class

$$\Delta(E) = c_2(E) - \frac{r-1}{2r} c_1(E)^2,$$

where $r = \operatorname{rk} E$.

Theorem 3.5. [21] *The following conditions are equivalent:*

(i) *E is semistable, and $\Delta(E) \cdot H^{n-2} = 0$;*
(ii) *for any morphism $f\colon C \to X$, where C is a smooth projective curve, the vector bundle $f^*(E)$ is semistable.*

The property in (ii), both for ordinary and Higgs bundles, will be called *curve semistability.*

Theorem 2 in [24] implies that the condition $\Delta(E) \cdot H^{n-2} = 0$ is equivalent, for a semistable bundle, to $\Delta(E) = 0$. In this form Theorem 3.5 was stated and proved with different techniques in [6]. It is now quite natural to ask if this theorem holds true also for Higgs bundles. As it was proved in [6, 5], one has:

Theorem 3.6. *A semistable Higgs bundle (E, ϕ) on (X, H) with $\Delta(E) = 0$ is curve semistable.*

The conjecture is that also the converse implication holds. Now, one easily shows that a curve semistable Higgs bundle is semistable with respect to any polarization. So the nontrivial content of the conjecture is the following:

Conjecture 3.7. *A curve semistable Higgs bundle has vanishing discriminant.*

We shall say that X is a *Higgs variety* if the conjecture holds on X.

By Theorem 3.6, a semistable Higgs bundle with vanishing Chern classes is curve semistable, and moreover has degree zero on any curve. As a result, it is H-numerically flat. This means that there is a morphism $\pi_1^H(X,x) \twoheadrightarrow \pi_1^{\mathrm{alg}}(X,x)$. Actually there is a commutative diagram of faithfully flat morphisms

$$\begin{tikzcd}
\pi_1^{\mathrm{alg}}(X,x) \arrow[r, two heads] & \pi_1^S(X,x) \\
& \pi_1^H(X,x) \arrow[ul, two heads] \arrow[u, two heads]
\end{tikzcd}.$$

In general, the groups $\pi_1^S(X,x)$ and $\pi_1^{\mathrm{alg}}(X,x)$ are different [13]; the previous diagram therefore shows that also the groups $\pi_1^H(X,x)$ and $\pi_1^S(X,x)$ are in general different.

Conjecture 3.7 can be reformulated in some alternative ways.

Theorem 3.8. *The following statements are equivalent.*

(i) *Conjecture 3.7 holds.*
(ii) *H-numerically flat Higgs bundles have vanishing Chern classes.*
(iii) *The natural morphism $\pi_1^H(X,x) \to \pi_1^{\mathrm{alg}}(X,x)$ is an isomorphism.*

Moreover, when the natural morphism $\pi_1^H(X,x) \to \pi_1^S(X,x)$ is an isomorphism, X is a Higgs variety.

Proof. Assume that the Conjecture 3.7 holds, and that $\mathfrak{E} = (E,\phi)$ is an H-numerically flat Higgs bundle. Then it is semistable; its pullback to any curve is H-numerically flat, hence semistable, so that $\mathfrak{E}$ is curve semistable. Then $\Delta(E) = c_2(E) = 0$. By Theorem 2 in [24], all Chern classes of E vanish, so that (ii) is proved.

Assume that (ii) holds, and let $\mathfrak{E} = (E,\phi)$ be a curve semistable Higgs bundle. By replacing it with its endomorphism bundle, we can assume it has vanishing first Chern class. Then its pullback to any curve has zero degree, and is semistable, so that it is H-nflat (see Lemma A.7 in [5] for details). But then $\mathfrak{E}$ itself is H-nflat, hence has vanishing Chern classes, which implies $\Delta(E) = 0$. So (i) and (ii) are equivalent.

On the other hand, (iii) is true if and only if the categories $\mathbf{HNF(X)}$ and $\mathbf{SH(X)}$ coincide, which amounts to saying that the Conjecture 3.7 holds.

Finally, if $\pi_1^H(X,x) \simeq \pi_1^S(X,x)$, then the categories $\mathbf{HNF(X)}$ and $\mathbf{NF(X)}$ coincide (namely, all H-nflat Higgs bundles have vanishing Higgs field), so that the property (ii) holds. □

In [8] some classes of Higgs varieties were identified.

(i) Varieties with nef tangent bundle (in dimension 2 and 3 they were classified in [10]).
(ii) Rationally connected varieties.
(iii) Fibrations over a Higgs variety whose fibres are rationally connected.
(iv) Bases of finite étale covers whose total space is a Higgs variety.
(v) Varieties of dimension ≥ 3 containing an effective ample divisor which is a Higgs variety.
(vi) The property of being a Higgs variety is a birational invariant.

Moreover, in [7] it was shown that the conjecture holds for algebraic K3 surfaces, i.e., K3 surfaces are Higgs varieties. Property (iv) implies that the same is true for Enriques surfaces over $\mathbb{C}$, which are quotients of a K3 surface by a free action of an order 2 group.

4. Examples

This Section is devoted to describe examples of the fundamental Higgs group of some varieties. Unfortunately, this group appears to be quite hard to compute, and at the moment the only way to have a grasp on it is to assume that the variety is Higgs, so that the Higgs fundamental group coincides with Simpson's algebraic fundamental group, and use the fact that the latter is the proalgebraic completion of the topological fundamental group.

X will always denote a smooth projective variety over the complex numbers.

4.1. **Simply connected Higgs varieties.** In this case all groups $\pi_1(X,x)$, $\pi_1^S(X,x)$, $\pi_1^H(X,x)$, $\pi_1^{\mathrm{alg}}(X,x)$ are trivial (indeed, $\pi_1^{\mathrm{alg}}(X,x)$ is the proalgebraic completion of the trivial group, so it is trivial too, together with $\pi_1^S(X,x)$; moreover, $\pi_1^H(X,x) \simeq \pi_1^{\mathrm{alg}}(X,x)$ as X is Higgs). Examples are provided by the rationally connected varieties [8] and K3 surfaces [7]. In this case, the categories $\mathbf{NF(X)}$, $\mathbf{HNF(X)}$ and $\mathbf{SH}(X)$ are all equivalent to the category $\mathbf{Vect}_{\Bbbk}$, which means that numerically effective vector bundles and H-nflat

Higgs bundles (i.e., semi-harmonic bundles) are all trivial, and the latter have zero Higgs field.

4.2. **Higgs varieties with finite fundamental group.** Let X be an Enriques surface over $\mathbb{C}$. As noted at the end of Section 3.3, it is a Higgs variety. Its fundamental group is $\mathbb{Z}_2$. From the definition of profinite completion we have $\hat{\mathbb{Z}}_2 = \mathbb{Z}_2$, and from Eq. (1) we also have $A(\mathbb{Z}_2) = \mathbb{Z}_2$. So we have

$$\pi_1(X,x) \simeq \pi_1^S(X,x) \simeq \pi_1^H(X,x) \simeq \pi_1^{\mathrm{alg}}(X,x) \simeq \mathbb{Z}_2 \,.$$

The categories $\mathbf{NF(X)}$, $\mathbf{HNF(X)}$ and $\mathbf{SH}(X)$ are all equivalent to the the category $\mathbf{Vect}_{\mathbb{C}}^{\mathbb{Z}_2}$ of $\mathbb{Z}_2$-graded vector spaces over $\mathbb{C}$ (for a subtlety about this category see [20]). All H-nflat Higgs bundles have zero Higgs field. Note that the pullback of an H-nflat bundle on X to Y (where Y is a K3 surface such that $Y/\mathbb{Z}_2 = X$) is H-nflat. The resulting functor $\mathbf{HNF(X)} \to \mathbf{HNF(Y)}$ is the forgetful functor $\mathbf{Vect}_{\mathbb{C}}^{\mathbb{Z}_2} \to \mathbf{Vect}_{\mathbb{C}}$ (the same for the other categories of bundles).

4.3. **Higgs varieties with free fundamental group.** Our supply of Higgs varieties essentially comes from varieties with nef tangent bundle (and varieties related to that by some easy geometric constructions, see again the final remarks in Section 3.3), with the only exception of K3 surfaces (and related varieties). Corollary 3.15 in [10] states that the fundamental group of a variety X with nef tangent bundle is an extension of $\mathbb{Z}^{2q}$ by a finite group, where q is the maximal irregularity of the finite étale covers of X. So for the moment we are only able to consider Higgs varieties with fundamental group having even rank.

We can in particular consider abelian varieties. In this case $\pi_1(X,x) = \mathbb{Z}^{2d}$, where $d = \dim X$. So we have (cf. Eq. (2))

$$\pi_1^S(X,x) \simeq \widehat{\mathbb{Z}^{2d}}, \qquad \pi_1^H(X,x) \simeq \mathbb{C}^{2d} \times T \times \widehat{\mathbb{Z}^{2d}}$$

with $\mathrm{char}(T) = (\mathbb{C}^*)^{2d}$. The morphism $\pi_1^H(X,x) \twoheadrightarrow \pi_1^S(X,x)$ is the projection of $\mathbb{C}^{2d} \times T \times \widehat{\mathbb{Z}^{2d}}$ onto its last factor. So in this case $\mathbf{NF}(X)$ is a proper subcategory of $\mathbf{HNF}(X)$.

References

[1] H. Bass, A. Lubotzky, A. Magid, and S. Mozes, *The proalgebraic completion of rigid groups*, Geom. Dedicata, 95 (2002), pp. 19–58.

[2] I. Biswas, U. Bruzzo, and S. Gurjar, *Higgs bundles and fundamental group schemes.* arXiv:1607.07207 [math.AG]. To appear in Adv. Geom.

[3] I. Biswas, A. J. Parameswaran, and S. Subramanian, *Monodromy group for a strongly semistable principal bundle over a curve*, Duke Math. J., 132 (2006), pp. 1–48.

[4] U. Bruzzo and B. Graña Otero, *Numerically flat Higgs vector bundles*, Commun. Contemp. Math., 9 (2007), pp. 437–446.

[5] ——, *Semistable and numerically effective principal (Higgs) bundles*, Adv. Math., 226 (2011), pp. 3655–3676.

[6] U. Bruzzo and D. Hernández Ruipérez, *Semistability vs. nefness for (Higgs) vector bundles*, Differential Geom. Appl., 24 (2006), pp. 403–416.

[7] U. Bruzzo, V. Lanza, and A. Lo Giudice, *Semistable Higgs bundles on Calabi-Yau manifolds.* arXiv:1710.03671 [math.AG].

[8] U. Bruzzo and A. Lo Giudice, *Restricting Higgs bundles to curves*, Asian J. Math., 20 (2016), pp. 399–408.

[9] P. Deligne and J. S. Milne, *Tannakian categories*, in Hodge cycles, motives, and Shimura varieties, P. Deligne, J. S. Milne, A. Ogus, and K.-Y. Shih, eds., vol. 900 of Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York, 1982, pp. ii+414.

[10] J.-P. Demailly, T. Peternell, and M. Schneider, *Compact complex manifolds with numerically effective tangent bundles*, J. Algebraic Geom., 3 (1994), pp. 295–345.

[11] A. Grothendieck and M. Raynaud, *Revêtements étales et groupe fondamental*, Séminaire de Géométrie Algébrique du Bois Marie – 1960-61 (SGA 1) (Documents Mathématiques 3), Société Mathématique de France, Paris, 1971–2003, pp. xviii+327. Exp. V, IX, X.

[12] R. Hartshorne, *Ample vector bundles*, Inst. Hautes Études Sci. Publ. Math., 29 (1966), pp. 63–94.

[13] A. Langer, *On the S-fundamental group scheme*, Ann. Inst. Fourier (Grenoble), 61 (2011), pp. 2077–2119.

[14] ——, *On the S-fundamental group scheme. II*, J. Inst. Math. Jussieu, 11 (2012), pp. 835–854.

[15] V. Lanza and A. Lo Giudice, *Bruzzo's conjecture*, J. Geom. Physics, 118 (2017), pp. 181–191.

[16] R. Lazarsfeld, *Positivity in algebraic geometry. I*, vol. 48 of Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[17] ——, *Positivity in algebraic geometry. II*, vol. 49 of Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.

[18] H. Lenstra, *Profinite groups.* Available at http://websites.math.leidenuniv.nl/algebra/Lenstra-Profinite.pdf.

[19] Y. Miyaoka, *The Chern classes and Kodaira dimension of a minimal variety*, in Algebraic geometry, Sendai, 1985, vol. 10 of Adv. Stud. Pure Math., North-Holland, Amsterdam, 1987, pp. 449–476.

[20] M. Müger, *Tensor categories: a selective guided tour,* Rev. Un. Mat. Argentina **51** (2010), pp. 95–163.

[21] N. Nakayama, *Normalized tautological divisors of semi-stable vector bundles*, Sūrikaisekikenkyūsho Kōkyūroku, (1999), pp. 167–173. Kyoto University, Research Institute for Mathematical Sciences.

[22] M. V. Nori, *On the representations of the fundamental group*, Compositio Math., 33 (1976), pp. 29–41.

[23] ———, *The fundamental group-scheme*, Proc. Indian Acad. Sci. Math. Sci., 91 (1982), pp. 73–122.

[24] C. T. Simpson, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math., 75 (1992), pp. 5–95.

[25] ———, *Local systems on proper algebraic V-manifolds*, Pure Appl. Math. Q., 7 (2011), pp. 1675–1759.